Operations on Sets

Conventionally, sets are denoted using curly brackets:

{Monday, Tuesday, Wednesday, Thursday, Friday}

$\{0\}$

{red, yellow, blue}

But listing the elements of a set is not practical except for sets of very small size.

Set comprehension

Most often, sets are defined by

- a *universe* of elements,
- a *predicate* on elements of the universe.

The set consists of those elements of the universe that *satisfy* the predicate.

Membership and Equality

$x \in S$ means x is a *member* of (or *element* of) set S.

$\in$ denotes the *membership relation*.

Two sets are equal if they have the same elements.

$$(S = T) \equiv \langle \forall x :: x \in S \equiv x \in T \rangle$$

Duality Between Sets and Predicates

$$[\ x \in \{y \mid p.y\} \quad \equiv \quad p.x\ ]$$

$$[\ \{x \mid x \in S\} \quad = \quad S\ ]$$

Set Union ($\cup$)

If S and T are sets (over the same universe)

$S \cup T$ is a set defined by

$$[\, x \in S \cup T \;\equiv\; x \in S \vee x \in T \,]$$

Duality tells us that $\cup$ inherits properties of $\vee$

$$[\, S \cup S = S \,]$$

$$[\, S \cup T = T \cup S \,]$$

$$[\, R \cup (S \cup T) = (R \cup S) \cup T \,]$$

Set Intersection ($\cap$)

If S and T are sets (over the same universe)

$S \cap T$ is a set defined by

$$[\, x \in S \cap T \;\equiv\; x \in S \wedge x \in T \,]$$

Intersection inherits properties of conjunction:

$$[\, S \cap S = S \,]$$

$$[\, S \cap T = T \cap S \,]$$

$$[\, R \cap (S \cap T) = (R \cap S) \cap T \,]$$

Union and intersection combined:

$$[\, R \cup (S \cap T) = (R \cup S) \cap (R \cup T) \,]$$
$$[\, R \cap (S \cup T) = (R \cap S) \cup (R \cap T) \,]$$

$$[\, R \cup (R \cap S) = R \,]$$
$$[\, R \cap (R \cup S) = R \,]$$

Set Complement

The complement of a set is the set of elements in its universe that are not in the set.

$$[\, x \in -S \quad \equiv \quad \neg(x \in S) \,]$$

($-S$ is sometimes denoted by $\overline{S}$).

Duality means that complement inherits the properties of (logical) negation.

$$[\, --S = S \,]$$

$$[\, -(S \cup T) = -S \cap -T \,]$$

$$[\, -(S \cap T) = -S \cup -T \,]$$

$$[\, (S = T) \equiv (-S = -T) \,]$$

Subset

A set S is a *subset* of set T if every element of S is an element of T.

$$[\, S \subseteq T \;\equiv\; \langle \forall x :: x \in S \Rightarrow x \in T \rangle \,].$$

$$[\, S \subseteq T \;\equiv\; S = S \cap T \,]$$

$$[\, S \subseteq T \;\equiv\; S \cup T = T \,]$$

Duality means that the subset relation inherits the properties of logical implication.

$$[\, S \subseteq S \,]$$

$$[\, R \subseteq T \;\Leftarrow\; R \subseteq S \wedge S \subseteq T \,]$$

$$[\, S = T \;\equiv\; (S \subseteq T) \wedge (T \subseteq S) \,]$$

$$[\, S \subseteq T \;\equiv\; -S \supseteq -T \,]$$

$$
\begin{array}{ll}
 & R \subseteq S \wedge S \subseteq T \\
= & \quad \{ \text{ definition } \} \\
 & R \cup S = S \wedge S \cup T = T \\
= & \quad \{ \text{ idempotence of } \wedge \} \\
 & R \cup S = S \wedge S \cup T = T \wedge S \cup T = T \\
= & \quad \{ \text{ Leibniz } \} \\
 & R \cup S = S \wedge S \cup T = T \wedge (R \cup S) \cup T = T \\
= & \quad \{ \text{ associativity } \} \\
 & R \cup S = S \wedge S \cup T = T \wedge R \cup (S \cup T) = T \\
= & \quad \{ \text{ Leibniz } \} \\
 & R \cup S = S \wedge S \cup T = T \wedge R \cup T = T \\
\Rightarrow & \quad \{ \text{ weakening } \} \\
 & R \cup T = T \\
= & \quad \{ \text{ definition } \} \\
 & R \subseteq T \, .
\end{array}
$$